GLOBAL BIFURCATION

We wish to use the degree to prove bifurcation result

We deal with maps

$$F(\lambda, x) = x - \lambda T x + R(x)$$

with T linear compact, $R \in C^1(X, X)$ compact, $R(0) = R'(0) = 0$

<u>NECESSARY CONDITION</u> $\quad d_x F(\lambda^*, 0)$ not invertible

$\iff \mathbb{1} - \lambda_* T$ not invertible

$\iff 0 \in \sigma(\mathbb{1} - \lambda_* T) \iff \frac{1}{\lambda_*} \in \sigma(T)$, $\lambda_* \neq 0$

<u>Thm (Krasnoselski)</u> X Banach reflexive, T compact operator, $R \in C^1(X, X)$ compact with $R(0) = 0$, $R'(0) = 0$

Let $\quad F(\lambda, x) = x - \lambda T x + R(x)$

If $\frac{1}{\lambda_*}$ is eigenv. of T with odd multiplicity, then

λ_* is bifurcation point

<u>proof</u> <u>BC</u> λ_* not bif. point. Then $\exists\, \varepsilon_0 > 0$ s.t. $\forall r \in (0, \varepsilon_0)$ and $\varepsilon \in (0, \varepsilon_0)$

(i) $F(\lambda, x) \neq 0 \quad \forall \lambda \in [\lambda_* - \varepsilon, \lambda_* + \varepsilon]$, $\forall \|x\| = r$ $\quad (*)$

(ii) $[\lambda_* - \varepsilon, \lambda_* + \varepsilon]$ does not contain λ s.t. $\frac{1}{\lambda}$ is eigenvalue of T

Define $\quad h(t, x) = x - \big((1-t)(\lambda_* - \varepsilon) + t(\lambda_* + \varepsilon)\big) T x + R(x)$

- $h(t, \cdot)$ compact perturb of identity
- $h(t, x)$ admissible on $B_r(0)$ by $(*)$

$$\Rightarrow \deg\big(F(\lambda_* - \varepsilon, \cdot), B_r, 0\big) = \deg\big(F(\lambda_* + \varepsilon, \cdot), B_r, 0\big)$$

$$\underset{\displaystyle i(F(\lambda_* - \varepsilon, \cdot), 0)}{\parallel} \qquad\qquad \underset{\displaystyle i(F(\lambda_* + \varepsilon, \cdot), 0)}{\parallel}$$

$$\begin{array}{ccc} \Vert & & \Vert \\ i(\mathbb{I}-(\lambda_*-\varepsilon)T,0) & & i(\mathbb{I}-(\lambda_*+\varepsilon)T,0) \\ \Vert & & \Vert \\ (-1)^{\beta_1} & & (-1)^{\beta_2} \end{array}$$

$$\beta_1 = \#\left(\text{e.g. of } T\right) > \frac{1}{\lambda_*-\varepsilon} \qquad \beta_2 = \left(\#\text{ eg of } T\right) > \frac{1}{\lambda_*+\varepsilon}$$

$$\Rightarrow \quad \beta_2 = \beta_1 + \underbrace{\text{multiplicity of } \frac{1}{\lambda_*}}_{\text{odd}}$$

$$\Rightarrow \quad \beta_2 \ \& \ \beta_1 \text{ different parity} \quad \lightning \qquad \square$$

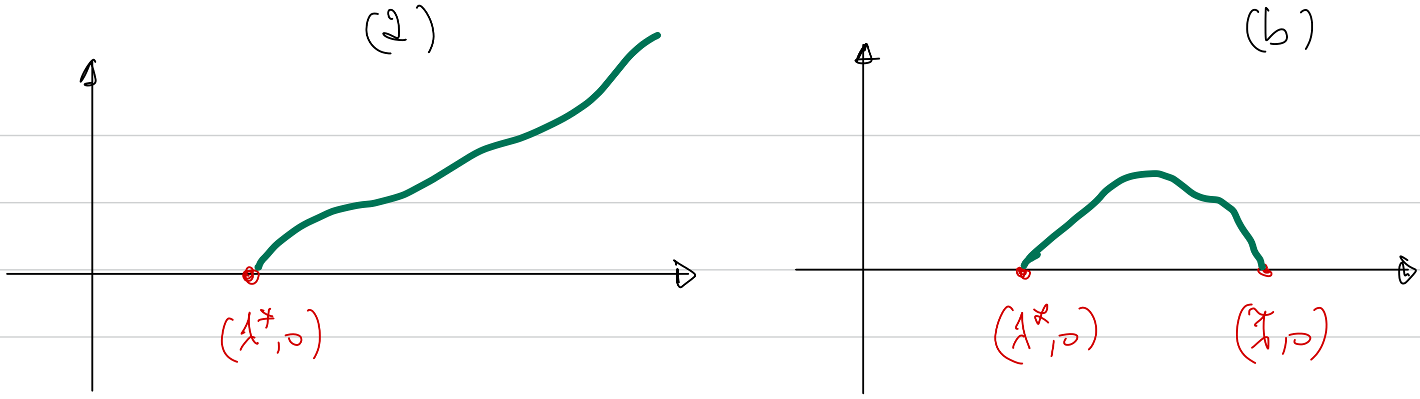
Rk The theorem shows that whenever the index change, there is a bifurcation

We conclude with a global bifurcation result, which shows what happens to local branches

$$\Sigma = \{ (\lambda, x) \in \mathbb{R} \times X \ , \ x \neq 0, \ F(\lambda, x) = 0 \}$$

$$F(\lambda, x) = x - \lambda T x + R(x) \quad , \quad R(0) = R'(0) = 0$$

Thm (Rabinowitz global bifurcation) Let T compact op, $R \in C^1(X, X)$ compact, $R(0) = R'(0) = 0$, $(\lambda^*, 0)$ bifurc point for $F(\lambda, x)$ s.t. $\frac{1}{\lambda^*}$ e.g of T with odd multiplicity
Let $\mathcal{C}$ be the connected component of Σ containing $(\lambda^*, 0)$. Then either (a) $\mathcal{C}$ is unbounded in $\mathbb{R} \times X$
or (b) $\exists\, \tilde{\lambda} \neq \lambda^*$ s.t. $(\tilde{\lambda}, 0) \in \mathcal{C}$

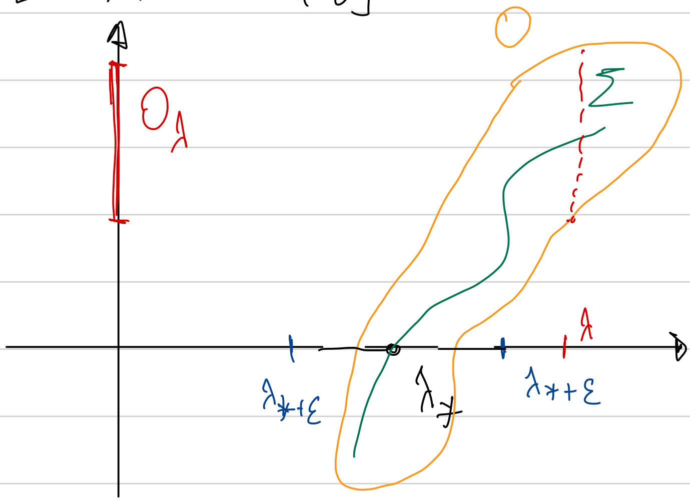


proof Assume that $\mathcal{C}$ does not fulfill (a) nor (b).
Then $\exists$ open bounded set $\mathcal{O}$ of $\mathbb{R}\times X$ containing $(\lambda^*,0)$ s.t.

(I) $\partial\mathcal{O}\cap\Sigma=\emptyset$

(II) $\exists\, \varepsilon>0$ s.t. $(\lambda,0)\in\mathcal{O} \Rightarrow |\lambda-\lambda_*|<\frac{\varepsilon}{2}$ and λ_* is the only bifurcation point $[\lambda_*-\varepsilon,\lambda_*+\varepsilon]$

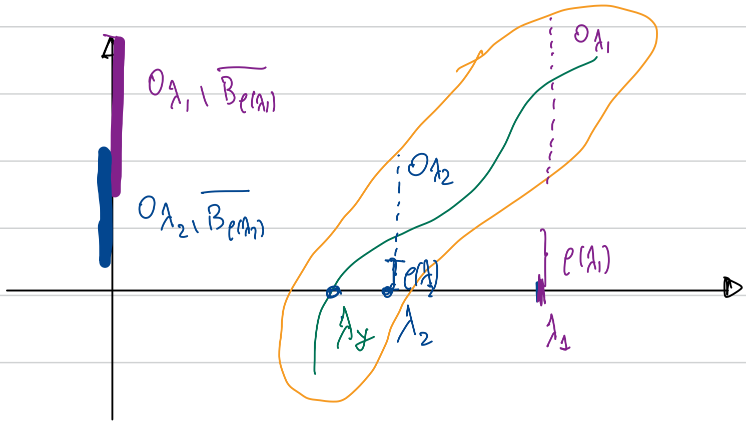
[AM, Lemma 4.6]



Put $\mathcal{O}_\lambda=\{x\in X : (\lambda,x)\in\mathcal{O}\}$

FACT if h adm. homotopy on $U\subset[a,b]\times X$, then $\deg(h(\lambda,\cdot),\mathcal{O}_\lambda,0)\equiv$ const in λ

proof [AM, Thm 4.1]



Given $(\lambda,0)$, $\lambda\neq\lambda_*$, we choose $\rho(\lambda)>0$ according to the rule

$$\rho(\lambda)<\begin{cases}\operatorname{dist}((\lambda,0),\overline{\mathcal{O}}) & \text{if } (\lambda,0)\notin\overline{\mathcal{O}}\\ \operatorname{dist}((\lambda,0),\Sigma) & \text{if } (\lambda,0)\in\overline{\mathcal{O}},\ \lambda\neq\lambda_*\end{cases}$$

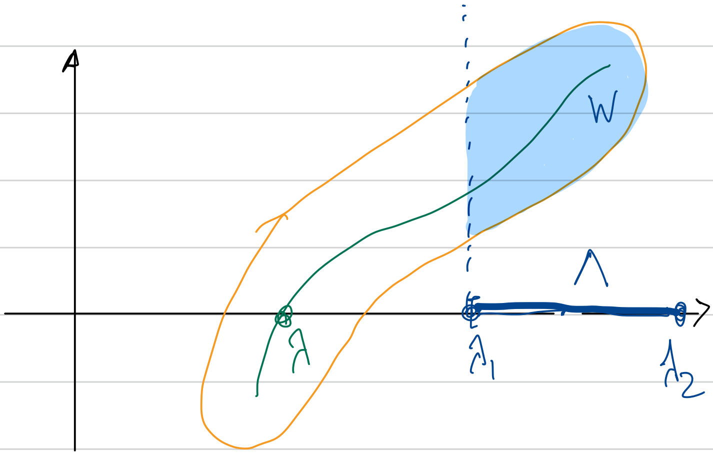
We shall consider $\forall\,\lambda\neq\lambda_*$ the set $\mathcal{O}_\lambda\setminus\overline{B_{\rho(\lambda)}}$

GOAL $\quad \deg\left(F(\lambda,\cdot),\ \mathcal{O}_\lambda \setminus \overline{B_{\rho(\lambda)}},\ 0\right) = 0 \quad \forall \lambda \neq \lambda_*$

To prove it, consider different regions of λ

•) if $|\lambda - \lambda_*| \gg 1$, true because $\mathcal{O}_\lambda = \emptyset$

•) Let $(\lambda_1, 0), (\lambda_2, 0) \notin \mathcal{O}$, $\lambda_1, \lambda_2 > \lambda_*$



Put $W := \mathcal{O} \cap (\Lambda \times X)$ open set

In this case $\mathcal{O}_\lambda \setminus \overline{B_{\rho(\lambda)}} = \mathcal{O}_\lambda$

$h(\lambda, x) = F(\lambda, x)$

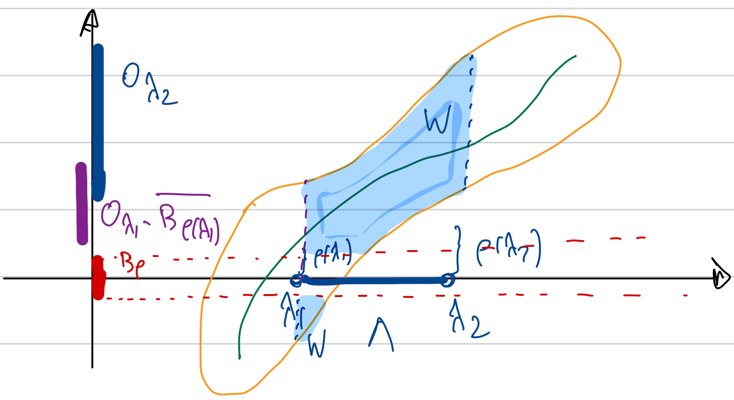
is admissible homotopy between $F(\lambda_1, x)$ & $F(\lambda_2, x)$

(indeed $h(\underline{\lambda}, \underline{x}) = 0$ for $\underline{x} \in \partial \mathcal{O}_\lambda$ implies $\partial \mathcal{O}_\lambda \cap \Sigma \neq \emptyset$ ↯)

hence $\quad \deg\left(F(\lambda_1,\cdot),\ \mathcal{O}_{\lambda_1},\ 0\right) = \deg\left(F(\lambda_2,\cdot), \mathcal{O}_{\lambda_2}, 0\right)$

by taking $|\lambda_2 - \lambda_*| \gg 1$ $\quad = 0$

•) Let $(\lambda_1, 0) \in \mathcal{O}$, $(\lambda_2, 0) \notin \mathcal{O}$, | $\lambda_1, \lambda_2 > \lambda_*$ and $[\lambda_1, \lambda_2] \not\ni$ other bifurcation points



Let $\rho < \rho(\lambda_1), \rho(\lambda_2)$ $\Rightarrow$ that

$(\Lambda \times \overline{B_\rho}) \cap \Sigma = \emptyset$

Put $W = \mathcal{O} \cap (\Lambda \times X \setminus \overline{B_\rho})$

$$\deg\left(F(\lambda_1,\cdot),\ \mathcal{O}_{\lambda_1} - \overline{B_\rho},\ 0\right) = \deg\left(F(\lambda_2,\cdot), \mathcal{O}_{\lambda_2} \setminus \overline{B_\rho}, 0\right)$$

$\|$ excision: no zeros in $\overline{B_{\rho(\lambda_1)}} - \overline{B_\rho}$ $\quad\quad \|$

$\deg\left(F(\lambda_1, \cdot),\ \mathcal{O}_{\lambda_1} - \overline{B_{\rho(\lambda_1)}},\ 0\right) \quad\quad 0$ previous point

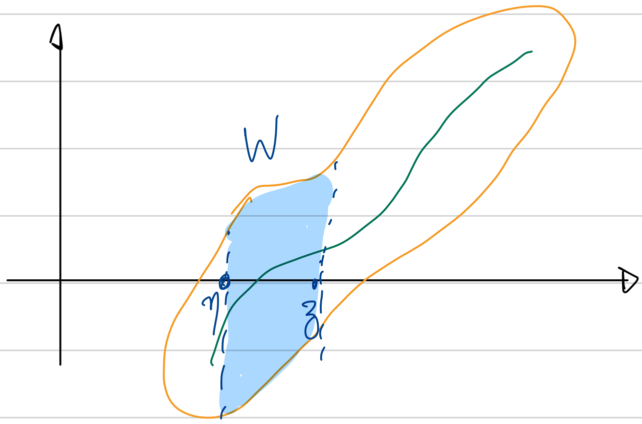
$$\Rightarrow \quad \deg\left(F(\lambda,\cdot),\ O_\lambda - \overline{B_{\rho(\lambda)}},\ 0\right) = 0 \quad \forall\ \lambda > \lambda_*$$

•) The same argument for $\lambda < \lambda_*$ shows that

$$\deg\left(F(\lambda,\cdot),\ O_\lambda \setminus \overline{B_{\rho(\lambda)}},\ 0\right) = 0 \quad \forall\ \lambda \neq \lambda_*$$

<u>CONTRADICTION</u> choose $\eta < \lambda_* < \zeta$ so that $(\eta,0), (\zeta,0) \in O$

Put $W = O \cap \left([\eta,\zeta] \times X\right)$



$$\deg\left(F(\eta,\cdot),\ O_\eta,\ 0\right)$$

$\|$ additivity property

$$\underbrace{\deg\left(F(\eta,\cdot),\ O_\eta \setminus \overline{B_{\rho(\eta)}},\ 0\right)}_{=0 \text{ by previous claim}} + \deg\left(F(\eta,\cdot),\ B_{\rho(\eta)},\ 0\right)$$

Similarly $\deg\left(F(\zeta,\cdot),\ O_\zeta,\ 0\right) = \deg\left(F(\zeta,\cdot),\ B_{\rho(\zeta)},\ 0\right)$

But, by the generalized homotopy property

$$\deg\left(F(\eta,\cdot),\ O_\eta,\ 0\right) = \deg\left(F(\zeta,\cdot),\ O_\zeta,\ 0\right)$$

$$\Rightarrow \quad \deg\left(F(\eta,\cdot),\ B_{\rho(\eta)},\ 0\right) = \deg\left(F(\zeta,\cdot),\ B_{\rho(\zeta)},\ 0\right)$$

$\|$ provided $\rho(\eta), \rho(\zeta)$ small enough $\|$

$$\deg\left(I - \eta T,\ B_{\rho(\eta)},\ 0\right) \qquad \deg\left(I - \zeta T,\ B_{\rho(\zeta)},\ 0\right)$$

$$\| \qquad\qquad\qquad \|$$

$$(-1)^{\beta(\eta)} \qquad\qquad (-1)^{\beta(\zeta)}$$

<u>BUT</u> $\beta(\eta), \beta(\zeta)$ diff parity since λ_* has multiplicity odd